x	y	z	a		x	y	z	a	
64	48	1599	1601	B	312	25	48984	48985	A
160	36	1677	1685	B					
60	11	1860	1861	A	308	75	50244	50245	A
					300	125	52812	52813	A
144	108	2021	2029	B					
16	63	2112	2113	A	364	27	66612	66613	A
56	33	2112	2113	A	360	81	68080	68081	A
96	28	2499	2501	B					
					420	29	88620	88621	A

Also solved in less detail by E. E. Whitford, C. E. Flanagan, and G. I. Hopkins.

GEOMETRY.

Note. The following remark should be made in connection with the solution of Geometry **417** in the September issue:

Three planes determine *either* a point (which may be at infinity if the intersections of the planes in pairs are parallel) *or* a straight line in which the planes are concurrent (which may be at infinity if the planes are parallel). Some of the 220 points required may therefore prove to be replaced by such lines of concurrency of three planes. Whenever the word *line* is used in this solution it refers to a straight line in which three planes are concurrent.

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers,

$$\int_0^1 |\cos 2\pi ax + \cos 2\pi bx|\,dx = \frac{2}{\pi ab}\left\{\frac{a+b}{\sin\dfrac{\pi}{a+b}} - \frac{a-b}{\sin\dfrac{\pi}{a-b}}\right\}$$

or

$$= \frac{1}{\pi ab}\left\{(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right\}$$

according as a and b are both odd or one of them is even.

Solution by the Proposer.

Take the first case with $a > b$.

The zeros of the integrand in the path of integration are

$$\xi_{1r} = \frac{2r-1}{2(a+b)}, \qquad r = 1, 3, 5, \cdots (a+b)$$

and

$$\xi_{2s} = \frac{2s-1}{2(a-b)}, \qquad s = 1, 3, 5, \cdots (a-b).$$

If these are $\xi_1, \xi_2, \xi_3, \cdots \xi_{2a}$ in order of magnitude the integral is

$$\frac{1}{\pi ab}\Big[b \sin 2\pi a\xi + a \sin 2\pi b\xi\Big]_{\xi_2,\ \xi_4,\ \cdots\ \xi_{2a}}^{\xi_1,\ \xi_3,\ \cdots\ \xi_{2a-1}}.$$

Consider first the sines of $2\pi b\xi_{1r}$. The rth is preceded by $r-1$ of its own set and by s of the other set where

$$s = \left[\frac{a-b}{2(a+b)}(2r-1)+\tfrac{1}{2}\right]; \ [\alpha] \equiv \text{greatest integer in } \alpha.$$

The expression is an integer only when $\xi_{2s} = \frac{1}{2}$ which does not occur in this case.

For the integral we have

$$\sin \pi \left\{ \frac{b(2r-1)}{2(a+b)} + (r-1) + \left[\frac{a-b}{2(a+b)}(2r-1)+\tfrac{1}{2}\right]\right\} = \sin \pi(L + [M]) \text{ say.}$$

Now $L + M = 2r - 1$ and M is not an integer. Hence $(L + [M]) = 2r - 2$ and the sine in question enters the integral positively. The set ξ_{1r} contains all odd multiples of $(a + b)/2$ and b is prime to $(a + b)$ and odd. The angles are then all odd multiples of $\pi/(a + b)$ and all being distinct and in the first two quadrants must be precisely the set

$$\pi k/(a+b), \quad k = 1, 3, 5, \cdots (a+b)-1.$$

The sum of this set of sines contributes to the integral

$$\frac{2}{\pi ab}\left(\frac{a}{\sin \pi/(a+b)}\right).$$

A similar count for the set $\sin 2\pi a\xi_{1r}$, using $L' - [M]$ instead of $L + [M]$, adds a term of the same form, a and b being interchanged.

Similarly the roots of the ξ_{2s} set with a contribute four times the sum of sines of all odd multiples less than the $(a - b)$th of $\pi/(a - b)$. With b the same set occurs with negative sign.

A similar method gives the formula in case 2.

350. Proposed by R. P. BAKER, University of Iowa.

Find a general formula for $d^n y/dx^n$ in terms of $d^k y/dt^k$ and $d^k x/dt^k$.

SOLUTION BY J. W. CLAWSON Collegeville, Pa.

Let δx, δy be increments of x, y when t takes the increment δt. Then, by Taylor's Theorem,

$$(1) \qquad \delta y = \delta x \frac{dy}{dx} + \frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2y}{dx^2} + \frac{\overline{\delta x}^3}{\underline{|3}}\frac{d^3y}{dx^3} + \cdots,$$

$$(2) \qquad \delta y = \delta \frac{dy}{dt} + \frac{\overline{\delta t}^2}{\underline{|2}}\frac{d^2y}{dt^2} + \frac{\overline{\delta t}^3}{\underline{|3}}\frac{d^3y}{dt^3} + \cdots,$$

$$(3) \qquad \delta t = \delta x \frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2} + \frac{\overline{\delta x}^3}{\underline{|3}}\frac{d^3t}{dx^3} + \cdots.$$

From (2) and (3) we get

$$(4) \quad \delta y = \left(\delta x \frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2} + \cdots\right)\frac{dy}{dt} + \left(\delta x \frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2} + \cdots\right)^2 \frac{1}{\underline{|2}}\frac{d^2y}{dt^2} + \cdots.$$